True and Apparent Moduli of Elasticity

1. True (Shear-Free) Modulus of Elasticity

In an engineering design, the deflection of a structural member under applied loads determines a service limit state that is often prescribed by the building code. In theory, the total deflection of a structural member subjected to design loads is composed of the deflections induced by bending and shear stresses:

$$\text{Total deflection } (\delta_T) = \text{Bending deflection } (\delta_b) + \text{Shear deflection } (\delta_s) \qquad \text{[Eq. 1]}$$

For example, the bending and shear deflections of a simply supported beam subjected to uniform loads can be calculated using Equations 2 and 3, respectively.

$$\delta_b = \frac{5w\ell^4}{384E_{true}I} \qquad \text{[Eq. 2]}$$

$$\delta_s = \frac{w\ell^2}{K} \qquad \text{[Eq. 3]}$$

where
w = applied uniform loads (lbf/in.),
ℓ = design span (in.),
E_{true} = true modulus of elasticity (lbf/in.^{2}),
I = moment of inertia (in.^{4}), and
K = shear deflection coefficient (lbf)

Since the bending deflection is calculated separately from the shear deflection, as shown in Equations 1 through 3, the modulus of elasticity used to calculate the bending component of the total deflection (Equation 2) shall be based on a shear-free modulus of elasticity, which is commonly termed "true E" in the wood engineering community.

Substituting Equations 2 and 3 into Equation 1, the total deflection of a simply supported beam subjected to uniform loads can be calculated as follows:

$$\delta_T = \frac{5w\ell^4}{384E_{true}I} + \frac{w\ell^2}{K} \qquad \text{[Eq. 4]}$$

As noted from Equations 2 and 3, the bending deflection is inversely proportional to the true modulus of elasticity, and the shear deflection is inversely proportional to the shear deflection coefficient of the structural member. Based on engineering mechanics, the shear deflection coefficient is proportional to the shear rigidity, G. For solid-sawn lumber, the shear rigidity is generally assumed to be 1/16 of the true modulus of elasticity. However, the E_{true}/G ratio could be significantly different from 16 for other engineered wood composites, such as structural composite lumber (SCL).

By assuming the E_{true}/G of 16, Equation 4 can be simplified to Equation 5 for bending members with a rectangular cross section.

$$\delta_T = \frac{5w\ell^4}{384E_{true}I} + \frac{12w\ell^2}{5E_{true}A}$$

[Eq. 5]

where A is the cross-sectional area.

Equation 5 can be further generalized with customary engineering units as follows:

$$\delta_T = \frac{270w\ell^4}{E_{true}bh^3} + \frac{28.8w\ell^2}{E_{true}bh}$$

[Eq. 6]

where δ_T = total deflection (in.),
  w = applied uniform loads (lbf/ft),
  ℓ = design span (ft),
  E_{true} = true modulus of elasticity (lbf/in.²),
  b = beam width (in.), and
  h = beam depth (in.)

2. Apparent Modulus of Elasticity

For prismatic bending members, the shear deflection under design loads is generally small, as compared to the bending deflection. Furthermore, the shear rigidity is an engineering parameter that is not commonly and precisely known for engineered wood products. Therefore, for simplicity and practicality, it is customary to use the so-called "apparent modulus of elasticity," $E_{apparent}$, for the calculation of total deflection as follows:

$$\delta_T = \frac{5w\ell^4}{384E_{apparent}I}$$

[Eq. 7]

The apparent modulus of elasticity is therefore related to the true modulus of elasticity by the following equation for a simply supported beam subjected to uniform loads:

$$E_{apparent} = \frac{E_{true}}{1 + \dfrac{384E_{true}I}{5K\ell^2}}$$

[Eq. 8]

It should be noted from Equation 8 that the relationship between $E_{apparent}$ and E_{true} depends on the span, moment of inertia, and shear deflection coefficient. For example, when the span becomes longer for the same bending member (I and K are the same), $E_{apparent}$ approaches E_{true}. On the other hand, when the span becomes shorter, the difference between $E_{apparent}$ and E_{true} becomes significant.

When compared with Equation 4, Equation 7 is a simpler equation that can be used in engineering design without requiring knowledge of an accurate shear rigidity value. For glulam products, ASTM D3737, *Standard Practice for Establishing Stresses for Structural Glued Laminated Timber (Glulam),* establishes the following relationship between $E_{apparent}$ and E_{true}:

$$E_{apparent} = 0.95 \times E_{true}$$

[Eq. 9]

Strictly speaking, the ratio of 0.95 between $E_{apparent}$ and E_{true} applies to a uniformly loaded simple beam with an E_{true}/G of 14 and a span-to-depth ratio of 21. The ratio decreases with increasing E_{true}/G ratio and decreasing span-to-depth ratio. For example, for a uniformly loaded simple beam with an E_{true}/G of 30 and a span-to-depth ratio of 18, which may be applicable to some SCL products, the ratio of $E_{apparent}$ and E_{true} can be calculated as 0.85.

3. Determination of E_{true} and $E_{apparent}$

The E_{true} and $E_{apparent}$ values for engineered wood products are typically determined through full-scale tests using a standard test method, such as ASTM D198, *Standard Test Methods of Static Tests of Lumber in Structural Sizes.* In general, if the mid-span deflection of a bending member is measured at the neutral axis by referencing to the reaction points, the modulus of elasticity determined from the test is an $E_{apparent}$ due to the fact that the shear deflection is already included in the experimental data. On the other hand, if the mid-span deflection of a bending member is measured at the neutral axis by referencing to the load points between which no shear stresses are induced, the modulus of elasticity determined from the test is an E_{true} due to the fact that the deflection is attributed to the bending stress without shear.

The design E values for glulam, as published by APA and most glulam manufacturers, are based on test results of $E_{apparent}$ unless otherwise noted. In this case, the designer can use Equation 7 in determining the total beam deflection with confidence.

4. Comparison between E_{true} and $E_{apparent}$

For most SCL products marketed in North America, such as laminated veneer lumber (LVL), the values of $E_{apparent}$ are generally published. However, some SCL manufacturers elect to publish E_{true}. Unfortunately, there are no standards in existence today requiring that the apparent modulus of elasticity be published for SCL. It is important to note that for those proprietary product manufacturers who publish E_{true}, the use of Equation 7 in determining the total deflection may be non-conservative due to the fact that the shear deflection has been neglected. With the increasing number of proprietary SCL products that are available in North America, if the designers do not realize that the published value is E_{true}, the design could be non-conservative when deflection governs the structural design.

The publication of $E_{apparent}$ and E_{true} values for a wide variety of engineered wood products without explaining which E value is being tabulated can be a cause of confusion when comparing similar products offered by different manufacturers or when the need for product substitution arises. To avoid confusion, Equation 9 can be used as a rule of thumb to convert a published E_{true} to $E_{apparent}$ so that a fair comparison between similar products can be made. For example, a product with an $E_{apparent}$ of 1.8 x 10^6 psi is generally equivalent to a product with an E_{true} of 1.9 x 10^6 psi. Similarly, a product with an $E_{apparent}$ of 2.1 x 10^6 psi is equivalent to a product with an E_{true} of 2.2 x 10^6 psi. For a precise conversion between E_{true} and $E_{apparent}$, please consult the manufacturer's literature.